Behavioural Analysis of the Modified Lotka–Volterra Predictor-Prey Model Using a Computer Simulation

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Outline

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Classical Predator-Prey Model ([2], [5], [3], [7])

$$\begin{aligned} \frac{dx}{dt} &= ax(t) - bx(t)y(t) \\ \frac{dy}{dt} &= -cy(t) + dx(t)y(t), \end{aligned} \tag{1}$$

- Population of Prey at time *t*: $x(t)$
- Population of Predator at time *t*: $y(t)$

Aim

Modify (1) to ensure asymptotic stability.

Problem Identification

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- The predators eat the prey and the prey live on the available natural food in the environment.
- Excessive consumption of prey reduces the food supply of the predators.
- A dwindling in predator’s population will result in a significant increase in the prey’s population.
- As the prey population increases, the food supply of the predator grows and so does the predators’ population.

Model Formulation and Assumptions

In this cyclic situation, one asks whether the cycle continues indefinitely or does one of the species eventually go into extinction.

Modification

Introduction of degree of internal competition of the prey and predator for their limited resources.

Assumptions

The degree of internal competition of the prey and predator are proportional to the square of the prey and predator populations respectively.

Proposed Model

Proposed Model

$$\begin{aligned} \frac{dx}{dt} &= ax(t) - bx(t)y(t) - ex^2(t) \\ \frac{dy}{dt} &= -cy(t) + dx(t)y(t) - fy^2(t), \end{aligned} \tag{2}$$

where a, b, c, d, e, f are positive constants.

Critical Points of Proposed Model

Critical Points of (2)

$$
\begin{aligned}
(x_1, y_1) &= \left(0, -\frac{c}{f}\right) \\
(x_2, y_2) &= \left(-\frac{-af - bc}{bd + ef}, -\frac{ce - ad}{bd + ef}\right) \\
(x_3, y_3) &= (0, 0) \\
(x_4, y_4) &= \left(\frac{a}{e}, 0\right)
\end{aligned} \tag{3}
$$

Eigenvalues of Proposed Model

Jacobian Matrix of (2)

$$
\begin{aligned}
J(x,y) &= \begin{pmatrix} \partial_x f(x,y) & \partial_y f(x,y) \\ \partial_x g(x,y) & \partial_y g(x,y) \end{pmatrix} \\
&= \begin{pmatrix} a - by - 2ex & -bx \\ dy & -c + dx - 2fy \end{pmatrix}
\end{aligned} \quad (4)
$$

Evaluating (4) respectively at the four critical points (3) results in

$$J_1(x,y) = \begin{pmatrix} a+\frac{bc}{f} & 0 \\ -\frac{cd}{f} & c \end{pmatrix}, \tag{5}$$

$$J_2(x,y) = \begin{pmatrix} -\frac{e(af+bc)}{bd+ef} & -\frac{b(af+bc)}{bd+ef} \\ \frac{d(ad-ce)}{bd+ef} & \frac{cef-adf}{bd+ef} \end{pmatrix} \tag{6}$$

Eigenvalues of Proposed Model

$$J_3(x,y) = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix} \tag{7}$$

$$J_4(x,y) = \begin{pmatrix} -a & -\frac{ab}{e} \\ 0 & \frac{ad}{e} - c \end{pmatrix} \tag{8}$$

The eigenvalues of (6), (5), (7) and (8) respectively are

$$\lambda_2 = \frac{1}{2(bd+ef)}\left(-bce - adf - aef + cef - \sqrt{\left((adf + aef + bce - cef)^2 - 4\left(a^2bd^2f + a^2def^2 + ab^2cd^2 - ace^2f^2 - b^2c^2de - bc^2e^2f\right)\right)}\right)$$

$$\mu_2 = \frac{1}{2(bd+ef)}\left(-bce - adf - aef + cef + \sqrt{\left((adf + aef + bce - cef)^2 - 4\left(a^2bd^2f + a^2def^2 + ab^2cd^2 - ace^2f^2 - b^2c^2de - bc^2e^2f\right)\right)}\right) \tag{9}$$

Eigenvalues of Proposed Model

$$\lambda_1 = c$$
$$\mu_1 = \frac{af + bc}{f} \tag{10}$$

$$\lambda_3 = a$$
$$\mu_3 = -c \tag{11}$$

$$\lambda_4 = -a$$
$$\mu_4 = \frac{ad - ce}{e} \tag{12}$$

Stability of Proposed Model

Eigenvalues	Linear System	Nonlinear System
$\lambda, \mu \in \mathbb{R}$		
$\lambda > \mu > 0$	Nodal Source (Unstable)	Nodal Source (Unstable)
$\lambda < \mu < 0$	Nodal Sink (Stable)	Nodal Sink (Stable)
$\lambda > 0 > \mu$	Saddle point (Unstable)	Saddle point (Unstable)
$\lambda = \mu > 0$	Degenerate Source or Nodal Source (Unstable) depending of the geometric multiplicity of λ	Source (Degenerate, Nodal, Spiral Source depending on the nonlinear terms)
$\lambda = \mu < 0$	Degenerate Sink or Nodal Sink (Stable) depending of	Sink (Degenerate, Nodal, Spiral Sink) (Stable de-

Stability of Proposed Model

Eigenvalues	Linear System	Nonlinear System
$\lambda, \mu \in \mathbb{C}$		
$Re(\lambda) > 0$	Spiral Source (Unstable)	Spiral Source (Unstable)
$Re(\lambda) < 0$	Spiral Sink (Stable)	Spiral Sink (Stable)
$Re(\lambda) = 0$	Center (Stable)	Center, Spiral Sink , Spiral Source (Stability cannot be determined based on λ)

Phase Portrait and Field Directions of the Model

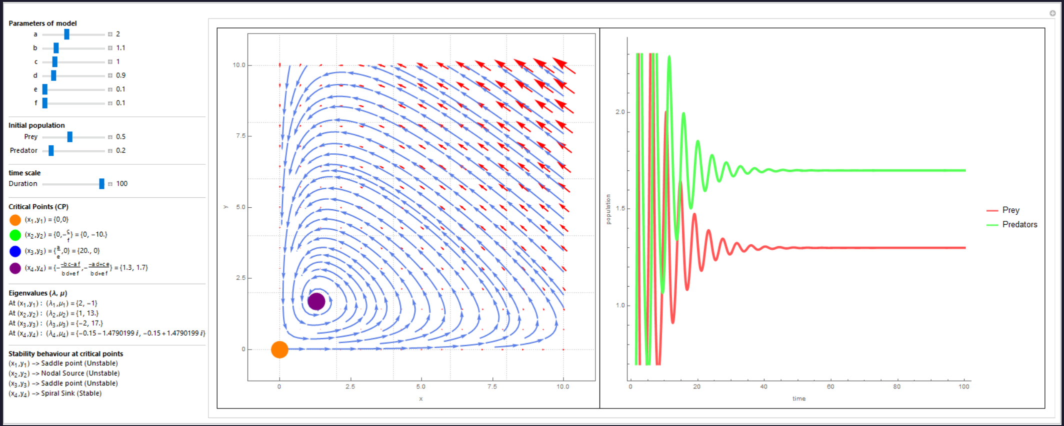

Figure 1: Phase portrait and population plots for (2) with $a = 2, b = 1.1, c = 1, d = 0.9, e = f = 0.1$

Phase Portrait and Field Directions of the Model

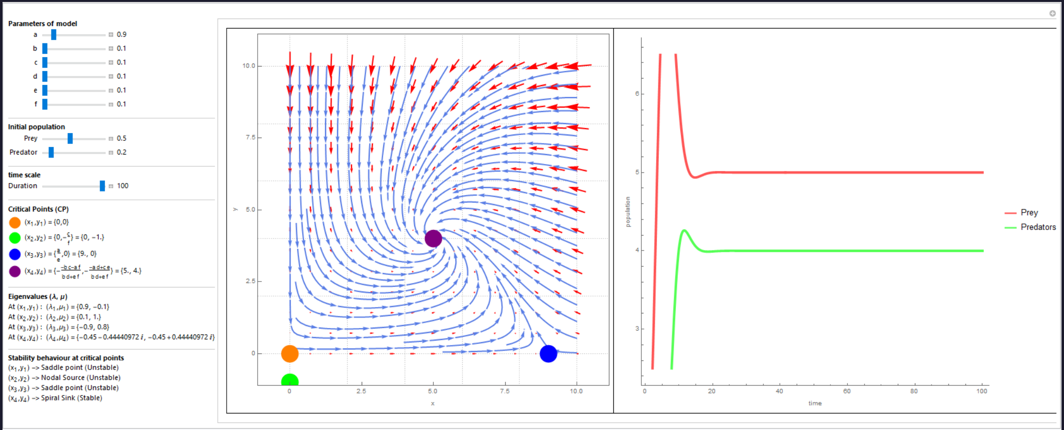

Figure 2: Phase portrait and population plots for (2) with $a = 0.9, b = 0.1, c = 0.1, d = 0.1, e = f = 0.1$

Phase Portrait and Field Directions of the Model

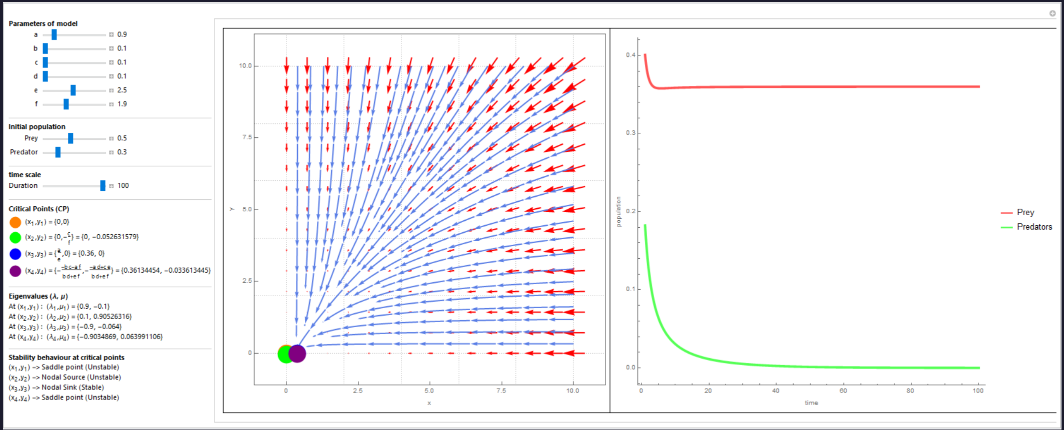

Figure 3: Phase portrait and population plots for (2) with $a = 0.9, b = 0.1, c = 0.1, d = 0.1, e = 2.5, f = 1.9$

Model Interpretation and Conclusion

Model Interpretation

From Figure 1, Figure 2 and Figure 3, it is clear that the stability at the various critical depends on the values of the model's parameters.

Model Interpretation

Based on the introduction of the degree of internal competition between the prey and the predators, we see from Figure 1, Figure 2 and Figure 3, that the trajectories of the model are not periodic and tend to equilibrium level with time.

Model Interpretation

Hence, the system (2) is asymptotically stable. This behaviour

Conclusion

The simulation reveals that neither the prey nor the predator will become extinct with time.

Thank You

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