On Pseudo-Automorphism of a Weak Inverse Property Loops

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Outlines

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Introduction

Abstract

The characterization of a W.I.P. loop \((Q, \cdot)\) is carried out in this investigation. This was done using the concept of pseudo-automorphism \(P\) with companion \(c\). \(Q\) is found to be an I.P. loop if every \(c \in Q\) is a companion of \(P\). The groups of all left, middle and right pseudo-automorphisms are also found to coincide in \(Q\). It is further established that, \(c\) satisfies kinds of generalized Moufang and Bol identities. It is also remarked that, if \((Q, \cdot)\) had been an inverse property, cross inverse property or commutative loops, the companion \(c\) of pseudo-automorphism \(P\) would have been Moufang. This effort extends the existing results in the literature in this direction.
Introduction

Definition of Quasigroups and Loops

- A set $Q$ with a binary operation $\cdot : Q \times Q \rightarrow Q$ is called a quasigroup if the equation $ax = xL_a = b$ and $ya = yR_a = b$ have unique solutions, $\forall x, y \in Q$ and $\forall a, b \in Q$.
- Note that $L_a$ and $R_a$ are known as translation maps for a fixed $a \in Q$ and we use the convention $ab := a \cdot b$.

Alternative Definition

- Equivalently, we can say that a groupoid is called a quasigroup if the maps $L_a : Q \rightarrow Q$ and $R_a : Q \rightarrow Q$ are bijections $\forall a \in Q$.
- If the quasigroup $Q$ contains an identity element, usually denoted by 1, then $Q$ is called a loop.
Motivation For the Study

- Desire to provide an alternative and easy to follow approach to the solutions in the literature.

- More importantly, since the ground breaking theorem that solved the long standing problem of finding a necessary and sufficient conditions for the isotopy-isomorphy property of loops was built around pseudo-automorphism, we found it expedient to look for an alternative solutions using the bijection in question.
Definition of Pseudo-automorphisms

1. A bijection $U$ on $(Q, \cdot)$ is called a right pseudo-automorphism of a quasigroup $Q$ if there exists at least one element $c \in Q$ such that
   
   $xU \cdot (yU \cdot c) = (xy)U \cdot c$.

2. A bijection $U$ on $(Q, \cdot)$ is called a middle pseudo-automorphism of a quasigroup $Q$ if there exists at least one element $c$ such that $[(xU)/(c \backslash 1)] \cdot [c \backslash (yU)] = (xy)U$. For all $x, y \in Q$, where the element $c$ is called a companion of $U$.

3. A bijection $U$ on $(Q, \cdot)$ is called a left pseudo-automorphism of a quasigroup $Q$ if there exists at least one element $c$ such that $(c \cdot xU) \cdot yU = c \cdot (xy)U$. For all $x, y \in Q$, where the element $c$ is called a companion of $U$. 
Definition of a Weak Inverse Property Loop (W.I.P.L.)

Definition

A loop \((Q, \cdot)\) with the identity element 1 is called a Weak Inverse Property loop (W.I.P. loop) if it satisfies the identical relation

\[ x \cdot (y \cdot x) J_\rho = y J_\rho \text{ or its dual } (x \cdot y) J_\lambda \cdot x = y J_\lambda \quad (2.1) \]

for all \(x, y \in Q\).
Introduction continues

Autotopisms of W.I.P.L

Theorem

Let \((Q, \cdot)\) be a W.I.P. loop with identity element 1. If \((U, V, W) \in ATP(Q, \cdot)\), then
\[(V, J_\lambda WJ_\rho, J_\lambda UJ_\rho), (J_\rho WJ_\lambda, U, J_\rho VJ_\lambda) \in ATP(Q, \cdot).\]

Moufang Loop

Definition

A loop \((Q, \cdot)\) with identity 1 is called a Moufang loop if it satisfies any of the Moufang identities
\[xy \cdot zx = (x \cdot yz)x,\]
\[xy \cdot zx = x(yz \cdot x),\]
\[(xy \cdot x)z = x(y \cdot xz)\] or \[(yx \cdot z)x = y(x \cdot zx),\]
for all \(x, y, z \in Q\).
Bol Loop and pseudo-automorphic concept.

Definition

A loop satisfying the right Bol identity \((xy \cdot z)y = x(yz \cdot y)\) for all \(x, y, z \in Q\) is called a Bol loop.

Definition

If \(P\) is a pseudo-automorphism of a loop \((Q, \cdot)\) with companion \(c\), then \(Q\) is called a pseudo-automorphic loop \((Q, \cdot)\) with companion \(c\).
Pseudo-automorphisms of W.I.P.

Propositions

Proposition

Let \((Q, \cdot)\) be a weak inverse property loop. Then \(P\) is a right pseudo-automorphism with companion \(c\) if and only if \(P\) is a middle pseudo-automorphism with companion \(c\).

Proposition

Let \((Q, \cdot)\) be a weak inverse property loop. Then \(P\) is a right pseudo-automorphism with companion \(c\) if and only if \(P\) is a left pseudo-automorphism of \((Q, \cdot)\) with companion \(cJ_\lambda\).
Corollary from the two Propositions

**Corollary**

Let \((Q, \cdot)\) be a weak inverse property loop. Then \(PSM_\lambda(Q, \cdot) = PSM_\mu(Q, \cdot) = PSM_\rho(Q, \cdot)\).

Emerging autotopisms

**Lemma**

Let \((Q, \cdot)\) be a pseudo-automorphic weak inverse property loop, with companion \(c\). Then

\[
Y = (L_{(cJ_\lambda)}, R_c^{-1}, R_c^{-1}L_{(cJ_\lambda)}) \quad (3.1)
\]

\[
T = (R_{(cJ_\rho)}, L_c R_c, R_c) \quad (3.2)
\]
Another Proposition

**Proposition**

Let \((Q, \cdot)\) be a pseudo-automorphic weak inverse property loop with companion \(c\). Then

\[
\phi_c = L_c L_{(cJ_\lambda)} = R_{(cJ_\rho)}^{-1} R_c^{-1} = L_c R_c L_c^{-1} R_c^{-1}
\]

is an automorphism of \((Q, \cdot)\).
Let \((Q, \cdot)\) be a pseudo-automorphic weak inverse property loop with companion \(c\). Then \((Q, \cdot)\) satisfies the identity

\[(cx) \cdot (y\phi_c \cdot c) = (c \cdot xy)c \quad \text{and} \quad (c \cdot x\phi_c) \cdot yc = c(xy \cdot c)\]

\((3.4)\)

\(\forall \ x, y \in Q\)
Pseudo-automorphisms of W.I.P.

A very good remarks

Remark

Observe that if the automorphism $\phi_c$ is an identity mapping, the identities (3.4) would have being a Moufang identity. That is, if $(Q, \cdot)$ had being inverse property, cross inverse property or commutative loop $c$ would have being straight Moufang element. The first identity in the Corollary 3.2 coincides with the generalized Moufang identity obtained by J.M. Osborn while carrying out study on a universal weak inverse loop via its loops isotopes. However, the second identity is appearing for the first time in the present work. It may be regarded as a dual to the first one.
Let \((Q, \cdot)\) be a pseudo-automorphic weak inverse property loop with companion \(c\). Then \((Q, \cdot)\) satisfies the identity

\[
[(x\phi_c \cdot c) \cdot y]c = x(cy \cdot c)
\]  

(3.5)

\(\forall x, y \in Q\)
The identity of the Corollary 3.3 would have being the right Bol identity if the automorphism $\phi_c = I$ the identity mapping. This identity may be called a generalized right Bol identity.


Thank You For Your Attentions!!!