On Pseudo-Automorphism of a Weak Inverse Property Loops

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Outlines

Introduction

Abstract

The characterization of a W.I.P. loop $(Q, \cdot)$ is carried out in this investigation. This was done using the concept of pseudo-automorphism P with companion c. Q is found to be an I.P. loop if every $c \in Q$ is a companion of P. The groups of all left, middle and right pseudo-automorphisms are also found to coincide in Q. It is further established that, c satisfies kinds of generalized Moufang and Bol identities. It is also remarked that, if $(Q, \cdot)$ had been an inverse property, cross inverse property or commutative loops, the companion c of pseudo-automorphism P would have been Moufang. This effort extends the existing results in the literature in this direction.

Introduction

Definition of Quasigroups and Loops

- A set Q with a binary operation $\cdot : Q \times Q \to Q$ is called a quasigroup if the equation $ax = xL_a = b$ and $ya = yR_a = b$ have unique solutions, $\forall\ x, y \in Q$ and $\forall\ a, b \in Q$.
- Note that L_a and R_a are known as translation maps for a fixed $a \in Q$ and we use the convention $ab := a \cdot b$.

Alternative Definition

- Equivalently, we can say that a groupoid is called a quasigroup if the maps $L_a{:}Q \to Q$ and $R_a{:}Q \to Q$ are bijections $\forall a \in Q$.
- If the quasigroup Q contains an identity element, usually denoted by 1, then Q is called a loop.

Introduction Cont.

Motivation For the Study

- Desire to provide an alternative and easy to follow approach to the solutions in the literature.
- More importantly, since the ground breaking theorem that solved the long standing problem of finding a necessary and sufficient conditions for the isotopy-isomorphy property of loops was built around pseudo-automorphism, we found it expedient to look for an alternative solutions using the bijection in question.

Preliminaries

Definition of Pseudo-automorphisms

1. A bijection U on $(Q,\cdot)$ is called a right pseudo-automorphism of a quasigroup Q if there exists at least one element $c \in Q$ such that
$xU \cdot (yU \cdot c) = (xy)U \cdot c$.
2. A bijection U on $(Q,\cdot)$ is called a middle pseudo-automorphism of a quasigroup Q if there exists at least one element c such that $[(xU)/(c\backslash 1)] \cdot [c\backslash(yU)] = (xy)U$. For all $x, y \in Q$, where the element c is called a companion of U.
3. A bijection U on $(Q,\cdot)$ is called a left pseudo-automorphism of a quasigroup Q if there exists at least one element c such that $(c \cdot xU) \cdot yU = c \cdot (xy)U$. For all $x, y \in Q$, where the element c is called a companion of U.

Preliminaries

Definition of a Weak Inverse Property Loop (W.I.P.L.)

Definition

A loop $(Q,\cdot)$with the identity element 1 is called a Weak Inverse Property loop (W.I.P. loop) if it satisfies the identical relation

$$x\cdot(y\cdot x)J_\rho = yJ_\rho \textit{ or its dual } (x\cdot y)J_\lambda\cdot x = yJ_\lambda \tag{2.1}$$

for all $x,y\in Q$.

Introduction continues

Autotopisms of W.I.P.L

Theorem

Let $(Q,\cdot)$be a W.I.P. loop with identity element 1*. If* $(U,V,W)\in ATP(Q,\cdot)$*, then* $(V,J_\lambda WJ_\rho,J_\lambda UJ_\rho),(J_\rho WJ_\lambda,U,J_\rho VJ_\lambda)\in ATP(Q,\cdot)$.

Moufang Loop

Definition

A loop $(Q,\cdot)$ *with identity* 1 *is called a Moufang loop if it satisfies any of the Moufang identities* $xy\cdot zx=(x\cdot yz)x$, $xy\cdot zx=x(yz\cdot x)$, $(xy\cdot x)z=x(y\cdot xz)$ *or* $(yx\cdot z)x=y(x\cdot zx)$*, for all* $x,y,z\in Q$.

Bol Loop and pseudo-automorphic concept.

Definition

A loop satisfying the right Bol identity $(xy \cdot z)y = x(yz \cdot y)$ *for all* $x, y, z \in Q$ *is called a Bol loop.*

Definition

If P *is a pseudo-automorphism of a loop* $(Q, \cdot)$*with companion* c*, then* Q *is called a pseudo-automorphic loop* $(Q, \cdot)$*with companion* c*.*

Pseudo-automorphisms of W.I.P.

Propositions

Proposition

Let $(Q, \cdot)$ be a weak inverse property loop. Then P is a right pseudo-automorphism with companion c if and only if P is a middle pseudo-automorphism with companion c.

Proposition

Let $(Q, \cdot)$be a weak inverse property loop. Then P is a right pseudo-automorphism with companion c if and only if P is a left pseudo-automorphism of $(Q, \cdot)$with companion cJ_λ.

Pseudo-automorphisms of W.I.P.

Corolary from the two Propositions

Corollary

Let $(Q,\cdot)$*be a weak inverse property loop. Then* $PSM_\lambda(Q,\cdot) = PSM_\mu(Q,\cdot) = PSM_\rho(Q,\cdot)$.

Emerging autotopisms

Lemma

Let $(Q,\cdot)$ *be an pseudo-automorphic weak inverse property loop, with companion c. Then*

$$Y = (L_{(cJ_\lambda)}, R_c^{-1}, R_c^{-1}L_{(cJ_\lambda)}) \quad (3.1)$$

$$T = (R_{(cJ_\rho)}, L_cR_c, R_c) \quad (3.2)$$

Pseudo-automorphisms of W.I.P.

Another Proposition

Proposition

Let $(Q,\cdot)$be a pseudo-automorphic weak inverse property loop with companion c. Then

$$\phi_c = L_c L_{(cJ_\lambda)} = R^{-1}_{(cJ_\rho)} R_c^{-1} = L_c R_c L_c^{-1} R_c^{-1} \qquad (3.3)$$

is an automorphism of $(Q,\cdot)$.

Pseudo-automorphisms of W.I.P.

Generalised identities

Corollary

Let $(Q,\cdot)$ be a pseudo-automorphic weak inverse property loop with companion c. Then $(Q,\cdot)$ satisfies the identity

$$(cx)\cdot(y\phi_c\cdot c)=(c\cdot xy)c \qquad and \qquad (c\cdot x\phi_c)\cdot yc=c(xy\cdot c) \tag{3.4}$$

$\forall\ x,y\in Q$

Pseudo-automorphisms of W.I.P.

A very good remarks

Remark

Observe that if the automorphism ϕ_c is an identity mapping, the identities (3.4) would have being a Moufang identity. That is, if $(Q, \cdot)$had being inverse property, cross inverse property or commutative loop c would have being straight Moufang element. The first identity in the Corollary 3.2 coincides with the generalized Moufang identity obtained by J.M. Osborn while carrying out study on a universal weak inverse loop via its loops isotopes. However, the second identity is appearing for the first time in the present work. It may be regarded as a dual to the first one.

Crypto-automorphism Cont.

Just One more Corollary

Corollary

Let $(Q, \cdot)$be a pseudo-automorphic weak inverse property loop with companion c. Then $(Q, \cdot)$ satisfies the identity

$$[(x\phi_c \cdot c) \cdot y]c = x(cy \cdot c) \tag{3.5}$$

$\forall\ x, y \in Q$

Crypto-automorphism Cont.

The last remarks

Remark

The identity of the Corollary 3.3 would have being the right Bol identity if the automorphism $\phi_c = I$ the identity mapping. This identity may be called a generalized right Bol identity.

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