Explicit Fourth-Derivative Two-Step Linear Multistep Method for Ordinary Differential Equations (ODEs)

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Problem Considered

The first-order initial value problem

$$\mathbf{u}' = \mathbf{f}(t, \mathbf{u}), \quad t \in [t_0, T], \quad \mathbf{u}(t_0) = \eta_{\mathbf{0}} \tag{1}$$

where $\mathbf{f} : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$, $\mathbf{u}, \eta_{\mathbf{0}} \in \mathbb{R}^n$ and $t_0, T \in \mathbb{R}$.

Form of Method to be Constructed

$$
\begin{aligned}
u_{n+2} - u_n &= \sum_{i=0}^{3} h^{i+1} \sum_{j=0}^{1} \beta_{ij} f_{n+j}^{(i)} \\
&= h\left(\beta_{00} f_n + \beta_{01} f_{n+1}\right) + \\
&\quad h^2\left(\beta_{10} f_n^{(1)} + \beta_{11} f_{n+1}^{(1)}\right) + \\
&\quad h^3\left(\beta_{20} f_n^{(2)} + \beta_{21} f_{n+1}^{(2)}\right) + \\
&\quad h^4\left(\beta_{30} f_n^{(3)} + \beta_{31} f_{n+1}^{(3)}\right) \qquad (2)
\end{aligned}
$$

Construction of Method

Associated Linear Difference Operator

$$\begin{aligned}\mathcal{L}[h,\gamma]u(t) &= u(t+2h) - u(t) - \\ &h\left(\beta_{00}u^{(1)}(t) + \beta_{01}u^{(1)}(t+h)\right) - \\ &h^2\left(\beta_{10}u^{(2)}(t) + \beta_{11}u^{(2)}(t+h)\right) - \\ &h^3\left(\beta_{20}u^{(3)}(t) + \beta_{21}u^{(3)}(t+h)\right) - \\ &h^4\left(\beta_{30}u^{(4)}(t) + \beta_{31}u^{(4)}(t+h)\right) \qquad (3)\end{aligned}$$

$$\gamma := (\beta_{00}, \beta_{01}, \beta_{10}, \beta_{11}, \beta_{20}, \beta_{21}, \beta_{30}, \beta_{31})$$

Corresponding Dimensionless Moment

$$L_m^*(\gamma) \;:=\; h^{-m}\mathcal{L}[h,\gamma]t^m|_{t=0} \tag{4}$$

Associated Algebraic System

Examining the algebraic system

$$L_m^*(\gamma) = 0, \quad m = 0, 1, 2, \cdots, M-1 \tag{5}$$

to find out the maximal *M* for which it is compatible.

Resulting Algebraic System from (5)

$$\left.\begin{aligned}
L_1^*(\gamma) &= -\beta_{00} - \beta_{01} + 2 = 0\\
L_2^*(\gamma) &= -2\left(\beta_{01} + \beta_{10} + \beta_{11} - 2\right) = 0\\
L_3^*(\gamma) &= -3\beta_{01} - 6\beta_{11} - 6\beta_{20} - 6\beta_{21} + 8 = 0\\
L_4^*(\gamma) &= -4\left(\beta_{01} + 3\beta_{11} + 6\beta_{21} + 6\beta_{30} + 6\beta_{31} - 4\right) = 0\\
L_5^*(\gamma) &= -5\beta_{01} - 4\left(5\beta_{11} + 15\beta_{21} + 30\beta_{31} - 8\right) = 0\\
L_6^*(\gamma) &= -2\left(3\beta_{01} + 15\beta_{11} + 60\beta_{21} + 180\beta_{31} - 32\right) = 0\\
L_7^*(\gamma) &= -7\beta_{01} - 42\beta_{11} - 210\beta_{21} - 840\beta_{31} + 128 = 0\\
L_8^*(\gamma) &= -8\left(\beta_{01} + 7\beta_{11} + 42\beta_{21} + 210\beta_{31} - 32\right) = 0\\
L_9^*(\gamma) &= -9\beta_{01} - 8\left(9\beta_{11} + 63\beta_{21} + 378\beta_{31} - 64\right) = 0.
\end{aligned}\right\} \qquad (6)$$

Construction of Method

Compatibility

The system (6) is compatible for the set

$$\{L_1^*(\gamma) = 0, L_2^*(\gamma) = 0, \cdots, L_8^*(\gamma) = 0\} \quad (7)$$

Maximal *M* for Compatibility

The maximal *M* for which (5) is compatible is *9*.

Classical Fitting Space

$$\left\{1, t, t^2, t^3, t^4, t^5, t^6, t^7, t^8, t^9\right\} \quad (8)$$

Solving (7) results in

Coefficients of Constructed Method

$$\left.\begin{array}{ll} \beta_{00} = 34, & \beta_{20} = \frac{20}{7}, \\ \beta_{01} = -32, & \beta_{21} = -\frac{80}{21}, \\ \beta_{10} = \frac{110}{7}, & \beta_{30} = \frac{22}{105}, \\ \beta_{11} = \frac{128}{7}, & \beta_{31} = \frac{16}{35} \end{array}\right\} \quad (9)$$

Nomenclature of Constructed Method (*FD2LMM*)

Fourth-Derivative Two-Step Explicit Linear Multistep Method

Constructed Method

$$\begin{aligned} u_{n+2} - u_n &= h\left(34 f_n - 32 f_{n+1}\right) + \\ &\quad \frac{1}{7} h^2 \left(110 f_n^{(1)} + 128 f_{n+1}^{(1)}\right) + \\ &\quad \frac{1}{21} h^3 \left(60 f_n^{(2)} - 80 f_{n+1}^{(2)}\right) + \\ &\quad \frac{1}{105} h^4 \left(22 f_n^{(3)} + 48 f_{n+1}^{(3)}\right) \end{aligned} \tag{10}$$

Lemma (Order of the *FD2LMM* method)

The FD2LMM method (10), and hence the associated operator L_m^* defined by (4) have order p if and only if

$$L_r^* \equiv 0, \quad r = 0, 1, \cdots, p, \quad L_{r+1}^* \not\equiv 0. \tag{11}$$

Qualitative Properties of Constructed Method

Theorem

The FD2LMM method (10) is of order 8.

Proof

Since with (9), $L_m^* = 0$ for $m = 0, 1, \cdots, 8$, and

$$L_9^* := -9\beta_{01} - 8\left(9\beta_{11} + 63\beta_{21} + 378\beta_{31} - 64\right), \quad (12)$$

substituting the coefficients (9) into (12) results in $L_9^* = \frac{736}{35} \neq 0$. Hence the *FD2LMM* method (10) is of order *8*.

The general expression of the leading term of the local truncation error (*lte*) for a method of the form (2) with order *p* (see [11]) can be written in the form:

$$lte(t) = (-1)^{p+1} h^{p+1} \frac{L^{*}_{p+1}}{(p+1)!} D^{p+1} u(t). \tag{13}$$

Qualitative Properties of Constructed Method

Local Truncation Error of the *FD2LMM* method (10)

Using (13), the local truncation error of *FD2LMM* method (10) is:

$$lte(t) = -h^9 \frac{736}{35.9!} u^{(9)}(t). \tag{14}$$

Theorem (Consistency)

The linear multistep method FD2LMM method (2) is said to be consistent if it has order $p \geq 1$*, [11].*

Lemma (Consistency of *FD2LMM* method (10))

The FD2LMM method (10) is consistent since it has order $p = 8 > 1$.

The first and second characteristics polynomials of the *FD2LMM* method (10) are respectively given by

First and Second Characteristics Polynomial

$$\rho(\xi) = \xi^2 - 1 \tag{15}$$

$$\sigma_i(\xi) = \sum_{j=0}^{1} \beta_{ij}\xi^j, \quad i = 0, 1, 2, 3. \tag{16}$$

Using (15) and (16), the stability polynomial of the *FD2LMM* method (10) is

Stability Polynomial of *FD2LMM* method (10)

$$\begin{aligned}\pi(\xi,\bar{h}) &= \rho(\xi)-\sum_{i=0}^{3}\bar{h}^{i+1}\sigma_i(\xi)\\ &= \frac{1}{105}\big(-48\bar{h}^4\xi-22\bar{h}^4+400\bar{h}^3\xi-300\bar{h}^3-\\ &\qquad 1920\bar{h}^2\xi-1650\bar{h}^2+3360\bar{h}\xi-\\ &\qquad 3570\bar{h}+105\xi^2-105\big) \qquad (17)\end{aligned}$$

Numerical Example 1

The nonlinear system from [13] and also studied in [10].

Problem 1

$$\begin{aligned} u_1'(t) &= -1002u_1 + 1000u_2; \quad u_1(0) = 1 \\ u_2'(t) &= u_2 - u_2(1+u_2); \quad u_2(0) = 1 \end{aligned} \tag{18}$$

Exact Solution

$$\begin{aligned} u_1(t) &= \exp(-2t) \\ u_2(t) &= \exp(-t) \end{aligned} \tag{19}$$

Numerical Results :: Problem 1

		Absolute Error			
		Wu-Xia		FD2LMM	
t	h	$u_1(t)$	$u_2(t)$	$u_1(t)$	$u_2(t)$
1	0.002	2.5606×10^{-07}	8.0150×10^{-08}	8.3267×10^{-17}	4.4409×10^{-16}
10	0.001	5.5468×10^{-16}	6.0936×10^{-12}	2.7756×10^{-17}	2.7756×10^{-16}

Table 1: Absolute errors of *"FD2LMM"* compared with *"Wu-Xia"* method, [13] at *t=1* and *t=10* on problem 1

Numerical Results :: Problem 1

		Absolute Error			
		SDAM		FD2LMM	
t	h	$u_1(t)$	$u_2(t)$	$u_1(t)$	$u_2(t)$
1	0.008	1.6348×10^{-14}	0.0000×10^{00}	1.1102×10^{-16}	0.0000×10^{00}
10	0.006	2.4815×10^{-24}	2.0329×10^{-20}	4.1359×10^{-24}	4.0658×10^{-20}

Table 2: Absolute errors of *"FD2LMM"* compared with *"SDAM"* method, [10] at *t=1* and *t=10* on problem 1

Numerical Example 2

The initial value problem considered in [4] on the range $0 \le t \le 1$.

Problem 2

$$
\begin{aligned}
u_1' &= -21u_1 + 19u_2 - 20u_3, \quad u_1(0) = 1 \\
u_2' &= 19u_1 - 21u_2 + 20u_3, \quad u_2(0) = 0 \\
u_3' &= 40u_1 - 40u_2 - 40u_3, \quad u_3(0) = -1.
\end{aligned}
$$

Numerical Example 2

Exact Solution

$$u(t)_1 = \frac{1}{2}e^{-40t}\left(e^{38t} + \sin(40t) + \cos(40t)\right)$$

$$u(t)_2 = -\frac{1}{2}e^{-40t}\left(e^{38t} - \sin(40t) - \cos(40t)\right)$$

$$u(t)_3 = -e^{-40t}(\cos(40t) - \sin(40t)).$$

Numerical Results :: Problem 2

	Relative Error				
	FD2LMM	SDAM	Amodio	SDAM	Amodio
Step	$k=2(p=8)$	$k=2(p=6)$	$k=5(p=6)$	$k=3(p=8)$	$k=7(p=8)$
20	1.5×10^{-7}	2.9×10^{-3}	5.7×10^{-2}	7.5×10^{-4}	2.9×10^{-2}
40	1.2×10^{-9}	7.3×10^{-5}	8.7×10^{-3}	1.9×10^{-5}	6.8×10^{-3}
80	7.2×10^{-12}	1.8×10^{-6}	4.9×10^{-4}	1.4×10^{-7}	7.8×10^{-5}
160	2.9×10^{-15}	3.3×10^{-8}	1.2×10^{-5}	6.4×10^{-10}	4.7×10^{-7}
320	2.3×10^{-15}	5.1×10^{-10}	2.2×10^{-7}	2.5×10^{-12}	2.3×10^{-9}
640	1.1×10^{-16}	7.7×10^{-12}	3.7×10^{-9}	9.8×10^{-15}	1.3×10^{-11}

Table 3: Relative errors of *"FD2LMM"* compared with the methods (SDAM)

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- A fourth-derivative two-step linear multistep method (FD2LMM) was constructed.
- The maximal order criteria was used for the construction.
- The (FD2LMM) method is consistent and has good stability property.
- The accuracy and efficiency of the (FD2LMM) method compared with methods in the literature is obvious from the numerical examples.

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Thank You