# Stability and Convergence of Two-Step Obrechkoff Scheme For Second-Order Two-Point Boundary Value Problem

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# Outline

## Problem Considered

$$u'' = f(t,u), \quad a < x < b, \quad u(a) = \eta_1, u(b) = \eta_2 \tag{1}$$

## Method Under Consideration

$$\begin{aligned} u_{n-1} - 2u_n + u_{n+1} = \\ \frac{1}{252}h^2\left(11f_{n-1} + 230f_n + 11f_{n+1}\right) - \\ \frac{1}{15120}h^4\left(13f_{n-1}^{(2)} - 626f_n^{(2)} + 13f_{n+1}^{(2)}\right) \end{aligned} \tag{2}$$

## Aim

*Establish conditions for convergence and stability of (2) when applied to (1).*

**Definition (Tridiagonal Matrice)**

A matrix $\mathbf{A} = (a_{ij})$ is tridiagonal if $a_{ij} = 0$, whenever $|i - j| > 1$.

**Definition (Irreducible Matrice)**

A tridiagonal matrix $\mathbf{A} = (a_{ij})$, is irreducible if and only if $a_{i,i-1} \neq 0,\ i = 2, 3, \cdot, N$ and $a_{i,i+1} \neq 0,\ i = 1, 2, \cdots, N - 1$

**Definition (Diagonally Dominant Matrice)**

A tridiagonal matrix $\mathbf{A} = (a_{ij})$, is diagonally dominant if

$$|a_{ii}| = \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}|, \quad i = 1, 2, \cdot, N$$

**Definition (Irreducibly Diagonally Dominant Matrice)**

A matrix $\mathbf{A} = (a_{ij})$, is said to be irreducibly diagonally dominant, if it is irreducible and diagonally dominant with inequality being satisfied for at least one *i*.

**Theorem**

*A matrix* $\mathbf{A} = (a_{ij})$, *is monotone if* $\mathbf{Az} \geq \mathbf{0} \Rightarrow \mathbf{z} \geq \mathbf{0}$.

The main properties of a monotone matrix are as follows:

- The monotone matrix $\mathbf{A}$ is nonsingular
- A matrix $\mathbf{A}$ is monotone if and only if $\mathbf{A}^{-1} \geq 0$

**Theorem**

*If a matrix* $\mathbf{A}$ *is irreducibly diagonally dominant and has nonpositive off-diagonal elements, then* $\mathbf{A}$ *is monotone*

Applying (2) to (1) yields the difference scheme

$$
\begin{aligned}
&-u_{n-1}+2u_n-u_{n+1}+\\
&\quad\frac{1}{252}h^2\left(11f_{n-1}+230f_n+11f_{n+1}\right)-\\
&\quad\quad\frac{1}{15120}h^4\left(13f^{(2)}_{n-1}-626f^{(2)}_{n}+13f^{(2)}_{n+1}\right)=0,\\
&\quad\quad\quad n=1,2,\cdots,N,
\end{aligned}
\tag{3}
$$

and the boundary condition becomes

$$
u_0=\eta_1,\quad u_{N+1}=\eta_2 \tag{4}
$$

From (3), the exact solution $u(t)$ of (1) satisfies

$$
\begin{aligned}
&-u(t_{n-1}) + 2u(t_n) - u(t_{n+1}) + \\
&\quad \frac{1}{252}h^2\left(11f\left(t_{n-1}, u\left(t_{n-1}\right)\right) + 230f\left(t_n, u\left(t_n\right)\right) + \right. \\
&\qquad \left. 11f\left(t_{n+1}, u\left(t_{n+1}\right)\right)\right) - \\
&\qquad \frac{1}{15120}h^4\left(13f^{(2)}\left(t_{n-1}, u\left(t_{n-1}\right)\right) - 626f^{(2)}\left(t_n, u\left(t_n\right)\right) + \right. \\
&\qquad \left. 13f^{(2)}\left(t_{n+1}, u\left(t_{n+1}\right)\right)\right) + \\
&\qquad\quad T_n = 0
\end{aligned} \tag{5}
$$

where $T_n$ is the truncation error.

Subtracting (5) from (3), applying the Mean Value Theorem and substituting $\varepsilon_n = u_n - u(t_n)$, the error equation is obtained as

**Error Equation**

$$\begin{aligned}&-\varepsilon_{n-1} + 2\varepsilon_n - \varepsilon_{n+1} + \\ &\quad \frac{1}{252}h^2\left(11\varepsilon_{n-1}f_{u_{n-1}} + 230\varepsilon_n f_{u_n} + 11\varepsilon_{n+1}f_{u_{n+1}}\right) - \\ &\quad\quad \frac{1}{15120}h^4\left(13\varepsilon_{n-1}f^{(2)}_{u_{n-1}} - 626\varepsilon_n f^{(2)}_{u_n} + 13\varepsilon_{n+1}f^{(2)}_{u_{n+1}}\right) - \\ &\quad\quad T_n = 0, \quad n = 1, 2, \cdots, N \end{aligned} \tag{6}$$

where the truncation error is given by

**Truncation Error**

$$T = \frac{59}{76204800}h^{10}u^{10}(\xi), \text{ and } \quad \|\mathbf{T}\| \leq \frac{59}{76204800}h^{10}\max_{\xi\in[a,b]}\left|u^{(10)}(\xi)\right| \tag{7}$$

In matrix notation, (6) can be written as

$$\mathbf{M}\mathbf{E} = \mathbf{T} \tag{8}$$

where

$$\mathbf{M} = \mathbf{J} + \mathbf{K} + \mathbf{L},$$

$$\mathbf{E} = [\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N]^T,$$

$$\mathbf{T} = [T_1, T_2, \cdots, T_N]^T$$

$$\mathbf{J} = \begin{pmatrix} 2 & -1 & & & \mathbf{0} \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & & \\ \mathbf{0} & & & -1 & 2 \end{pmatrix} \tag{9}$$

$$\mathbf{K} = \frac{h^2}{252}\begin{pmatrix} 230f_{u_1} & 11f_{u_2} & & & \mathbf{0} \\ 11f_{u_1} & 230f_{u_2} & 11f_{u_3} & & \\ & 11f_{u_2} & 230f_{u_3} & 11f_{u_4} & \\ & & \ddots & & \\ \mathbf{0} & & & 11f_{u_{N-1}} & 230f_{u_N} \end{pmatrix} \tag{10}$$

# Convergence Analysis

$$\mathbf{L} = \frac{h^4}{15120}\begin{pmatrix} 626f_{u_1}^{(2)} & -13f_{u_2}^{(2)} & & & \mathbf{0} \\ -13f_{u_1}^{(2)} & 626f_{u_2}^{(2)} & -13f_{u_3}^{(2)} & & \\ & -13f_{u_2}^{(2)} & 626f_{u_3}^{(2)} & -13f_{u_4}^{(2)} & \\ & & \ddots & & \\ \mathbf{0} & & & -13f_{u_{N-1}}^{(2)} & 626f_{u_N}^{(2)} \end{pmatrix} \tag{11}$$

Since

$$f_{u_n} > 0, \; n = 1, 2, \cdots N$$

then

$$\mathbf{K} + \mathbf{L} \geq \mathbf{0}, \quad \text{and} \quad \mathbf{M} = \mathbf{J} + \mathbf{K} + \mathbf{L} \geq \mathbf{J}.$$

Clearly, $\mathbf{J}$ is monotone.

**Off-diagonal elements of $\mathbf{M}$**

$$-1+\frac{11}{252}h^2 f_{u_{n-1}}-\frac{13}{15120}h^4 f^{(2)}_{u_{n-1}} \quad (12)$$

and

$$-1+\frac{11}{252}h^2 f_{u_{n+1}}-\frac{13}{15120}h^4 f^{(2)}_{u_{n+1}} \quad (13)$$

**Diagonal elements of $\mathbf{M}$**

$$2+\frac{230}{252}h^2 f_{u_n}+\frac{626}{15120}h^4 f^{(2)}_{u_n} \quad (14)$$

For $\mathbf{M}$ to have non-negative off diagonal elements, *h* must be chosen such that both (12) and (13) are respectively non-negative.

## Choice of $h$

$$-1+\frac{11}{252}h^2 f_u-\frac{13}{15120}h^4 f_u^{(2)}<0 \quad \text{over} \;\; [a,b] \qquad (15)$$

and

$$2+\frac{230}{252}h^2 f_{u_n}+\frac{626}{15120}h^4 f_{u_n}^{(2)} \geq \left|-1+\frac{11}{252}h^2 f_{u_{n-1}}-\frac{13}{15120}h^4 f_{u_{n-1}}^{(2)}\right|+\left|-1+\frac{11}{252}h^2 f_{u_{n+1}}-\frac{13}{15120}h^4 f_{u_{n+1}}^{(2)}\right| \qquad (16)$$

$\mathbf{M}$ is irreducibly diagonally dominant and monotone following (15) and (16)

Since $\mathbf{M} \geq \mathbf{J}$, we have that $\mathbf{0} < \mathbf{M}^{-1} \leq \mathbf{J}^{-1}$.

From (8),

$$\begin{aligned}
\mathbf{E} &= \mathbf{M}^{-1}\mathbf{T} \\
\Rightarrow \|\mathbf{E}\| &\leq \|\mathbf{M}^{-1}\|\|\mathbf{T}\| \leq \|\mathbf{J}^{-1}\|\|\mathbf{T}\| \\
&\leq \left(\frac{(b-a)^2}{8h^2}\right)\left(\frac{59}{76204800}h^{10}\max_{\xi\in[a,b]}\left|u^{(10)}(\xi)\right|\right) \\
&= \frac{59(b-a)^2}{609638400}h^{8}\max_{\xi\in[a,b]}\left|u^{(10)}(\xi)\right| \qquad (17)
\end{aligned}$$

It follows that the method is of order 8 and

$$\lim_{h\to 0}\|\mathbf{E}\| = 0 \quad \text{or} \quad \lim_{h\to 0} u_j = u(t_j) \qquad (18)$$

This establishes the convergence of (2).

**Theorem**

*A method with stability function* $R_{mm}(\lambda^2)$ *has an interval of periodicity* $(0, \lambda_0^2)$ *if* $\left|R_{mm}(\lambda^2)\right| < 1$ *for* $0 < \lambda^2 < \lambda_0^2$.

**Theorem**

*A method with stability function* $R_{mm}(\lambda^2)$ *is P-stable if* $\left|R_{mm}(\lambda^2)\right| < 1$ *for all real* $\lambda \neq 0$.

Applying (2) to the test problem

$$u'' = -k^2 u, \tag{19}$$

results in

$$u_{n-1} - 2u_n + u_{n+1} = -\frac{1}{252}\lambda^2 \left(11u_{n-1} + 230u_n + 11u_{n+1}\right) - \frac{1}{15120}\lambda^4 \left(13u_{n-1} - 626u_n + 13u_{n+1}\right) \tag{20}$$

where $\lambda = kh$.

Rearranging and simplifying (20) gives

$$u_{n-1} - 2R_{22}u_n + u_{n+1} = 0,$$

where

$$R_{22}(\lambda^2) = \frac{1 - \frac{115}{252}\lambda^2 + \frac{313}{15120}\lambda^4}{1 + \frac{11}{252}\lambda^2 + \frac{13}{15120}\lambda^4}. \tag{21}$$

The rational expression (21) is the stability function of (2).

## Plot of Stability Function

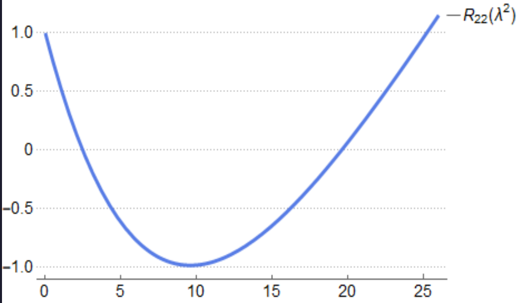


Figure 1: The stability function $R_{22}$ as a function of $\lambda^2$

From the above, it is clear that the (2) is not *P-stable* but has a large interval of periodicity, i.e $[0, 25.2]$

Using some properties of matrices, the necessary conditions for the two-step Obrechkoff method (2) to be convergent has been established.

It has been shown that the method is not *P-stable* but has a large interval of periodicity.