Construction and Implementation of Optimal 8–Step Linear Multistep method

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Outline

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First Order Initial Value Problem

$$y' = f(x, y), \quad x_0 < x < b, \quad y(x_0) = y_0, \tag{1}$$

General *k*-step Linear Multistep Method for (1)

$$\sum_{j=0}^{k} \alpha_j y_{n+j} \;=\; h \sum_{j=0}^{k} \beta_j y'_{n+j}, \tag{2}$$

where α_j and β_j are uniquely determined.

Form of Method to be Constructed

$$\sum_{i=0}^{k=8} \alpha_i y_{n+i} = h \sum_{i=0}^{k=8} \beta_i y'_{n+i}. \tag{3}$$

First Characteristics Polynomial of (3)

$$\rho(\xi) = \sum_{i=0}^{k=8} \alpha_i \xi^i \tag{4}$$

Optimality Condition for (3)

All the roots of (4) must lie on the unit circle in the complex plane.

Consistency of (3)

With $\xi = 1$ as a root of $\rho(\xi) = 0$, we have that

$$\sum_{i=0}^{k=8} \alpha_i \; = \; 0, \tag{5}$$

which established the consistency of (3).

Since (4) is a polynomial of degree 8, it must have another real root on the unit circle.

Zero-Stability of (3)

This other real root must be *-1* and the remaining *6* roots must be complex.

General form of the roots of (4)

$$\left.\begin{array}{ll} \xi_1 = +1, & \xi_2 = -1, \\ \xi_3 = e^{i\theta_1}, & \xi_4 = e^{-i\theta_1}, \\ \xi_5 = e^{i\theta_2}, & \xi_6 = e^{-i\theta_2}, \\ \xi_7 = e^{i\theta_3}, & \xi_8 = e^{-i\theta_3} \end{array}\right\}, \qquad 0 < \theta_1, \theta_2, \theta_3 < \pi \tag{6}$$

With (6), we have

$$\rho(\xi) = (\xi-1)(\xi+1)\left(\xi-e^{i\theta_1}\right)\left(\xi-e^{-i\theta_1}\right)\left(\xi-e^{i\theta_2}\right)\times \left(\xi-e^{-i\theta_2}\right)\left(\xi-e^{i\theta_3}\right)\left(\xi-e^{-i\theta_3}\right), \quad (7)$$

$$\begin{aligned}\rho(\xi) &= \xi^8+\xi^7\left(-2\cos(\theta_1)-2\cos(\theta_2)-2\cos(\theta_3)\right)+\\ &\xi^6\left(4\cos(\theta_1)\cos(\theta_2)+4\cos(\theta_3)\cos(\theta_2)+4\cos(\theta_1)\cos(\theta_3)+2\right)+\\ &\xi^5\left(-8\cos(\theta_2)\cos(\theta_3)\cos(\theta_1)-2\cos(\theta_1)-2\cos(\theta_2)-2\cos(\theta_3)\right)+\\ &\xi^3\left(8\cos(\theta_2)\cos(\theta_3)\cos(\theta_1)+2\cos(\theta_1)+2\cos(\theta_2)+2\cos(\theta_3)\right)+\\ &\xi^2\left(-4\cos(\theta_1)\cos(\theta_2)-4\cos(\theta_3)\cos(\theta_2)-4\cos(\theta_1)\cos(\theta_3)-2\right)+\\ &\xi\left(2\cos(\theta_1)+2\cos(\theta_2)+2\cos(\theta_3)\right)-1. \quad (8)\end{aligned}$$

Setting $\cos(\theta_1) = \lambda_1$, $\cos(\theta_2) = \lambda_2$ and $\cos(\theta_3) = \lambda_3$ in (8)

and comparing the coefficients of powers of ξ in the resulting expression with (4), yields

$$\left.\begin{aligned}
\alpha_8 &= +1, \\
\alpha_7 &= -2\left(\lambda_1 + \lambda_2 + \lambda_3\right), \\
\alpha_6 &= 4\lambda_2\lambda_3 + 4\lambda_1\left(\lambda_2 + \lambda_3\right) + 2, \\
\alpha_5 &= -2\left(\lambda_2 + \lambda_3 + \lambda_1\left(4\lambda_2\lambda_3 + 1\right)\right), \\
\alpha_4 &= 0, \\
\alpha_3 &= 2\left(\lambda_2 + \lambda_3 + \lambda_1\left(4\lambda_2\lambda_3 + 1\right)\right), \\
\alpha_2 &= -2\left(2\lambda_2\lambda_3 + 2\lambda_1\left(\lambda_2 + \lambda_3\right) + 1\right), \\
\alpha_1 &= 2\left(\lambda_1 + \lambda_2 + \lambda_3\right), \\
\alpha_0 &= -1
\end{aligned}\right\} \tag{9}$$

Optimal Order Requirements (see [7])

The order requirements with *t=4*, [7]) in terms of the coefficients gives (8).

Comparing the coefficients of powers of ξ with (4), yields

$$\beta_0+\beta_1+\beta_2+\beta_3+\beta_4+\beta_5+\beta_6+\beta_7+\beta_8 = -16\left(\lambda_1-1\right)\left(\lambda_2-1\right)\left(\lambda_3-1\right)$$

$$4\beta_0+3\beta_1+2\beta_2+\beta_3=\beta_5+2\beta_6+3\beta_7+4\beta_8$$

$$3\left(16\beta_0+9\beta_1+4\beta_2+\beta_3+\beta_5+4\beta_6+9\beta_7+16\beta_8\right) = -16\left(7\lambda_3+\lambda_2\left(7-4\lambda_3\right)+\lambda_1\left(\lambda_2\left(\lambda_3-4\right)-4\lambda_3+7\right)-10\right)$$

$$64\beta_0+27\beta_1+8\beta_2+\beta_3=\beta_5+8\beta_6+27\beta_7+64\beta_8$$

$$1280\beta_0+405\beta_1+80\beta_2+5\beta_3+5\beta_5+80\beta_6+405\beta_7+1280\beta_8 = 16\left(-61\lambda_3+\lambda_2\left(16\lambda_3-61\right)+\lambda_1\left(-\lambda_2\left(\lambda_3-16\right)+16\lambda_3-61\right)+136\right)$$

$$1024\beta_0 + 243\beta_1 + 32\beta_2 + \beta_3 = \beta_5 + 32\beta_6 + 243\beta_7 + 1024\beta_8$$

$$28672\beta_0 + 5103\beta_1 + 448\beta_2 + 7\beta_3 + 7\beta_5 + 448\beta_6 + 5103\beta_7 + 28672\beta_8 =$$
$$16\left(-547\lambda_3 + \lambda_2\left(64\lambda_3 - 547\right) + \lambda_1\left(-\lambda_2\left(\lambda_3 - 64\right) + 64\lambda_3 - 547\right) + 2080\right)$$

$$16384\beta_0 + 2187\beta_1 + 128\beta_2 + \beta_3 = \beta_5 + 128\beta_6 + 2187\beta_7 + 16384\beta_8$$

$$589824\beta_0 + 59049\beta_1 + 2304\beta_2 + 9\beta_3 + 9\beta_5 + 2304\beta_6 + 59049\beta_7 + 589824\beta_8 =$$
$$16\left(-4921\lambda_3 + \lambda_2\left(256\lambda_3 - 4921\right) + \right.$$
$$\left.\lambda_1\left(-\lambda_2\left(\lambda_3 - 256\right) + 256\lambda_3 - 4921\right) + 32896\right)$$

$$262144\beta_0 + 19683\beta_1 + 512\beta_2 + \beta_3 = \beta_5 + 512\beta_6 + 19683\beta_7 + 262144\beta_8$$

Solving the above system of equation, we get

$$\beta_0 = \frac{\lambda_1(188+52\lambda_3+\lambda_2(52+23\lambda_3))+2(1991+94\lambda_3+\lambda_2(94+26\lambda_3))}{14175}$$

$$\beta_1 = -\frac{2(-11552+4922\lambda_3+\lambda_2(4922+448\lambda_3)+\lambda_1(4922+448\lambda_3+\lambda_2(448+167\lambda_3)))}{14175}$$

$$\beta_2 = \frac{4(1819-9484\lambda_3+\lambda_2(-9484+5494\lambda_3)+\lambda_1(-9484+5494\lambda_3+\lambda_2(5494+701\lambda_3)))}{14175}$$

$$\beta_3 = \frac{\lambda_1(-27268+\lambda_2(70528-46378\lambda_3)+70528\lambda_3)+4(19312-6817\lambda_3+\lambda_2(-6817+17632\lambda_3))}{14175}$$

$$\beta_4 = -\frac{2(-358+\lambda_2(7708-4348\lambda_3)+7708\lambda_3+\lambda_1(7708-4348\lambda_3+\lambda_2(-4348+13903\lambda_3)))}{2835}$$

$$\beta_5 = \frac{\lambda_1(-27268+\lambda_2(70528-46378\lambda_3)+70528\lambda_3)+4(19312-6817\lambda_3+\lambda_2(-6817+17632\lambda_3))}{14175}$$

$$\beta_6 = \frac{4(1819-9484\lambda_3+\lambda_2(-9484+5494\lambda_3)+\lambda_1(-9484+5494\lambda_3+\lambda_2(5494+701\lambda_3)))}{14175}$$

$$\beta_7 = -\frac{2(-11552+4922\lambda_3+\lambda_2(4922+448\lambda_3)+\lambda_1(4922+448\lambda_3+\lambda_2(448+167\lambda_3)))}{14175}$$

$$\beta_8 = \frac{\lambda_1(188+52\lambda_3+\lambda_2(52+23\lambda_3))+2(1991+94\lambda_3+\lambda_2(94+26\lambda_3))}{14175}$$

Following [7], the expression for the error constant (lte) associated with this method is given by

Local Truncation Error

$$lte = \frac{-4\left(305\lambda_3 + \lambda_2\left(124\lambda_3 + 305\right) + 1246\right)}{935550} - \frac{\lambda_1\left(496\lambda_3 + \lambda_2\left(263\lambda_3 + 496\right) + 1220\right)}{935550} \quad (10)$$

The choice of values of λ_1, λ_2, and λ_3

$\lambda_1 = \frac{1}{\sqrt{2}}$, $\lambda_2 = 0$, $\lambda_3 = -\frac{1}{\sqrt{2}}$

The above choice makes (3) to be zerostable.

Constructed Method

$$y_{n+8} - y_n = \frac{3956}{14175} f_n + \frac{23552}{14175} f_{n+1} - \frac{3712}{14175} f_{n+2} + \frac{41984}{14175} f_{n+3} - \frac{3632}{2835} f_{n+4} + \frac{41984}{14175} f_{n+5} - \frac{3712}{14175} f_{n+6} + \frac{23552}{14175} f_{n+7} + \frac{3956}{14175} f_{n+8} \qquad (11)$$

Since method (11) is zerostable and its order *p=10 > 1*, then it is convergent.

We shall refer to the new scheme as (*Opt8sM*).

Problem 1

$$y'(x) = x + y, \quad y(0) = 1 \qquad x \in [0, 1]$$

with exact solution

$$y(x) = 2\exp(x) - x - 1;$$

Numerical Results :: Problem 1

t_n	$Y_{exact}(t_n)$	$E8RK(t_n)$	$Opt8sM(t_n)$	Abs. Error $E8RK(t_n)$	Abs. Error $Opt8sM(t_n)$
0.8	2.6510819	2.6510819	2.6510819	7.2495787E-11	2.1316726E-11
0.9	3.0192062	3.0192062	3.0192062	9.0135899E-11	2.4826807E-11
1.0	3.4365637	3.4365637	3.4365637	1.1068435E-10	3.8390624E-11

Table 1: The absolute error of the proposed *Opt8sM* method compared with *(E8RK)* with steplength h=0.1

Numerical Results :: Problem 1

t_n	$Y_{exact}(t_n)$	$E8RK(t_n)$	$Opt8sM(t_n)$	Abs. Error $E8RK(t_n)$	Abs. Error $Opt8sM(t_n)$
0.5000	1.7974425	1.7974425	1.7974425	2.1573854E-12	4.3032244E-13
0.5625	1.9476093	1.9476093	1.9476093	2.5839331E-12	5.6310512E-13
0.6250	2.1114919	2.1114919	2.1114919	3.0562219E-12	9.1393559E-13
0.6875	2.2899749	2.2899749	2.2899749	3.5784709E-12	9.7699626E-13
0.7500	2.4840000	2.4840000	2.4840000	4.1553427E-12	1.4583890E-12
0.8125	2.6945696	2.6945696	2.6945696	4.7921667E-12	1.6253665E-12
0.8750	2.9227506	2.9227506	2.9227506	5.4933835E-12	2.0223823E-12
0.9375	3.1696789	3.1696789	3.1696789	6.2660988E-12	2.4273916E-12
1.0000	3.4365637	3.4365637	3.4365637	7.1151973E-12	1.0853540E-12

Problem 2

$$y'(x) = x^2 y, \quad y(0) = 1 \qquad x \in [0, 1]$$

with exact solution

$$y(x) = \exp\left(\frac{1}{3}x^3\right)$$

t_n	$Y_{exact}(t_n)$	$E8RK(t_n)$	$Opt8sM(t_n)$	Abs. Error $E8RK(t_n)$	Abs. Error $Opt8sM(t_n)$
0.40	1.021563	1.021563	1.021563	2.384759E-13	1.317613E-12
0.45	1.030841	1.030841	1.030841	3.397283E-13	1.762591E-12
0.50	1.042547	1.042547	1.042547	4.691803E-13	2.304823E-12
0.55	1.057025	1.057025	1.057025	6.317169E-13	3.035128E-12
0.60	1.074655	1.074655	1.074655	8.333334E-13	3.981260E-12
0.65	1.095862	1.095862	1.095862	1.080913E-12	5.300427E-12
0.70	1.121126	1.121126	1.121126	1.382228E-12	7.093659E-12
0.75	1.150993	1.150993	1.150993	1.746159E-12	9.588108E-12
0.80	1.186095	1.186095	1.186095	2.182032E-12	1.414402E-11

- An optimal 8-step linear multistep method for first-order differential equations has been constructed.
- The constructed method is consistent and zero-stable. Hence it is convergent.
- The accuracy of the method compared with the well-known Runge-Kutta method is demonstrated by its application to two test problems.

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